Sharper Responses, Less Stable Adjustment: Synchronization in a Binary Congestion Game

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***Abstract*—Automated decision systems can improve each decision while making population adjustment less stable when many users revise together. This paper isolates that tension in a nonatomic binary congestion game. Agents choose through a logit response with precision β, and a deterministic share λ revises in each batch. The game has a unique logit fixed point that is independent of λ, but its stability is not. We prove that equilibrium response gain increases strictly with precision whenever congestion creates an interior payoff crossing. The fixed point is geometrically stable exactly when $\lambda[1+\beta k q_\beta(1-q_\beta)] < 2$, yielding an explicit synchronization budget. Above the full-batch threshold, a nondegenerate period-two orbit exists. In contrast, the independent-clock mean dynamic is globally stable at every finite precision. Thus the revision protocol, rather than decision quality alone, determines whether sharper response improves or destabilizes aggregate behavior.**

***Index Terms*—congestion games, logit response, population games, synchronization, dynamic stability**

I. Introduction

A route planner, recommendation model, trading rule, or purchasing agent can become better at ranking actions while producing a worse aggregate trajectory. If an action's payoff declines with the number of agents taking it, then a common, sharp response sends many agents toward the same action. The action becomes unattractive, so the next batch reverses. Static optimization can therefore coexist with dynamic overshoot.

The motivating concern is often described as algorithmic monoculture: many decision makers may rely on the same scoring technology, increasing common exposure even when the score is individually accurate [1]. Related feedback appears when predictions alter the distributions they predict [2]. These broad settings combine common information, correlated errors, heterogeneous objectives, and institutional constraints. This paper instead solves one narrow question: in a symmetric binary congestion game, how do response precision and the revision protocol jointly determine whether the unique logit fixed point is approached or overshot?

The question belongs to a long line of congestion and population games [3], [4], quantal response equilibrium [5], and stochastic strategy revision [6]. Iterated logit response can leave a quantal-response fixed point dynamically unstable [7], while concurrent and one-at-a-time logit updates can have different stochastic behavior [8]. Accordingly, neither logit response nor update-dependent stability is claimed as a new equilibrium concept. The contribution is a deliberately small benchmark for automated population decisions: it derives the exact batch-revision stability boundary in primitive parameters, proves that the boundary tightens monotonically with response precision under an interior congestion crossing, separates partial batch revision from genuine independent-clock revision, and exhibits a synchronized two-cycle above the full-batch threshold.

Three statements that sound similar are mathematically different. A more precise response can move a static fixed point toward the noiseless congestion equilibrium. A smaller batch can damp a discrete update without changing that fixed point. Independent revision clocks instead generate a continuous mean flow that is globally stable in the same model. Precision, batch size, and clock speed should not be collapsed into a single notion of capability.

II. Model

Consider a unit mass of agents. Each chooses action 1, a congestible resource, or action 0, an outside option. Let $q \in [0,1]$ denote the share choosing action 1. Its payoff advantage is

$$D(q) = b - kq, \qquad k > 0, \qquad 0 < b < k. \tag{1}$$

Here b is the uncongested advantage and k is congestion sensitivity. The inequalities imply an interior payoff crossing. The noiseless Nash or Wardrop share is $r = b/k \in (0,1)$.

Agents use the logistic choice function

$$F_\beta(q) = \sigma(\beta[b-kq]), \qquad \sigma(z) = \frac{1}{1+e^{-z}}, \tag{2}$$

where $\beta \geq 0$ is response precision. At $\beta = 0$, each action is selected with probability one half. As β increases, the higher-payoff action receives more probability. Precision is a behavioral parameter, not a complete measure of intelligence.

Time is discrete. In each period a deterministic, representative fraction $\lambda \in (0,1]$ revises, while the rest retain their action. The mean-field batch map is

$$q_{t+1} = G_{\beta,\lambda}(q_t) = (1-\lambda)q_t + \lambda F_\beta(q_t). \tag{3}$$

The parameter λ is the batch revision share. It is not itself clock speed or genuine asynchrony. Equation (3) is exact for the nonatomic representative-batch model and a deterministic mean approximation for a large finite population.

Lemma 1 (Unique logit fixed point): For every finite $\beta \geq 0$ and $\lambda \in (0,1]$, (3) has one fixed point $q_\beta \in (0,1)$. It does not depend on λ and satisfies

$$q_\beta = F_\beta(q_\beta) \iff \log\frac{q_\beta}{1-q_\beta} = \beta(b-kq_\beta). \tag{4}$$

Proof: Let $h(q) = F_\beta(q) - q$. Since $h(0) > 0 > h(1)$, continuity gives a zero. Moreover, $h'(q) = -\beta k F_\beta(q)[1 -$

$F_\beta(q)] - 1 < 0$, so the zero is unique. Because $\lambda > 0$, $G_{\beta,\lambda}(q) = q$ if and only if $F_\beta(q) = q$. ∎

The finite-precision point q_β differs from the noiseless share r. It lies between $1/2$ and r and converges to r as $\beta \to \infty$. Define the dimensionless equilibrium response gain

$$m(\beta) = \beta k q_\beta (1 - q_\beta). \tag{5}$$

Lemma 2 (Precision raises equilibrium gain): Under (1), $m(0) = 0$, $m(\beta) \to \infty$, and $m(\beta)$ is strictly increasing. Hence a unique $\beta_c > 0$ satisfies $m(\beta_c) = 1$.

Proof: Write $\alpha = \beta k$, $r = b/k$, and $s = q_\beta(1 - q_\beta)$. Implicit differentiation of (4) gives

$$\frac{dq_\beta}{d\alpha} = \frac{s(r - q_\beta)}{1 + \alpha s}. \tag{6}$$

Differentiating $m = \alpha s$ and substituting (6) yields

$$\frac{dm}{d\alpha} = \frac{s}{1 + \alpha s}\left[1 + \alpha\{(q_\beta - r)^2 + r(1 - r)\}\right] > 0. \tag{7}$$

Equation (4) implies $q_\beta \to r$. Since $r(1 - r) > 0$, $m = \alpha q_\beta(1 - q_\beta) \to \infty$. Continuity and monotonicity give the unique crossing. ∎

The interior crossing is necessary. If $b > k$, action 1 strictly dominates, $q_\beta \to 1$, and $m(\beta) \to 0$; sharper response eventually stabilizes rather than destabilizes the map.

III. Stability Under Batch Revision

The static fixed point is invariant to the batch share, but its dynamic stability is not.

Theorem 1 (Exact batch stability boundary): Under (1),

$$G'_{\beta,\lambda}(q_\beta) = 1 - \lambda[1 + m(\beta)]. \tag{8}$$

The fixed point is locally geometrically asymptotically stable exactly when

$$\boxed{\lambda[1 + m(\beta)] < 2}. \tag{9}$$

It is unstable when the left side exceeds two. At equality it is locally asymptotically stable but not geometrically stable.

Proof: Differentiating (3) and using $F_\beta(q_\beta) = q_\beta$ gives (8). This derivative is below one and lies in $(-1, 1)$ exactly under (9), which is necessary and sufficient for local geometric stability of this differentiable one-dimensional map. Above the boundary it is below -1, proving instability.

At equality the linear test is inconclusive. Let $x = q - q_\beta$ and expand $G(q_\beta + x) - q_\beta = -x + ax^2 + cx^3 + O(x^4)$, where $a = \lambda F''_\beta(q_\beta)/2$ and $c = \lambda F'''_\beta(q_\beta)/6$. Composition gives

$$G^2(q_\beta + x) - q_\beta = x - 2(a^2 + c)x^3 + O(x^4). \tag{10}$$

With $s = q_\beta(1 - q_\beta)$ and $\lambda = 2/(1 + m)$, direct differentiation of (2) gives

$$a^2 + c = \frac{\lambda(\beta k)^2 m}{6(1 + m)}[-1 + 6s + 2m(1 - 3s)] > 0. \tag{11}$$

Indeed $0 < s \le 1/4$ and equality requires $m \ge 1$; the bracket is increasing in m and equals one at $m = 1$. Thus two iterations reduce every sufficiently small nonzero $|x|$. Convergence is nongeometric because $|G'(q_\beta)| = 1$. ∎

The theorem yields the synchronization budget

$$\lambda < \bar{\lambda}(\beta) = \frac{2}{1 + m(\beta)}. \tag{12}$$

When $m(\beta) < 1$, every feasible batch share is geometrically stable. When $m(\beta) > 1$, full-batch revision is unstable and stability requires a smaller batch. Lemma 2 implies that $\bar{\lambda}(\beta)$ decreases strictly after entering the feasible interval.

For the centered case $b = k/2$, symmetry fixes $q_\beta = 1/2$, so $m(\beta) = \beta k/4$. Defining $\gamma = \beta k$, the exact boundary becomes

$$\lambda < \frac{8}{\gamma + 4}. \tag{13}$$

Full-batch revision therefore loses geometric stability at $\gamma = 4$.

Corollary 1 (Alternating orbit): If $\lambda = 1$ and $m(\beta) > 1$, a nondegenerate period-two orbit exists.

Proof: Let $J(q) = F_\beta^2(q) - q$. Then $J(0) > 0$, $J(1) < 0$, $J(q_\beta) = 0$, and $J'(q_\beta) = m(\beta)^2 - 1 > 0$. Hence J is negative immediately left of q_β, so continuity gives a root $q_L \in (0, q_\beta)$. Strict decrease of F_β implies $q_H = F_\beta(q_L) > q_\beta$ and $F_\beta(q_H) = q_L$. Thus (q_L, q_H) is a nondegenerate two-cycle. The argument proves existence, not uniqueness or global attraction. ∎

IV. Revision Protocols and Numerical Check

Partial batch revision should not be mislabeled as genuine asynchrony. If agents instead receive independent revision opportunities in continuous time, the deterministic population limit is

$$\dot{q} = F_\beta(q) - q. \tag{14}$$

Corollary 2 (Independent-clock stability): For every finite β, (14) converges globally to q_β.

Proof: The right side is strictly decreasing, has the unique zero q_β, is positive below it, and is negative above it. Every nonstationary trajectory moves monotonically toward q_β. ∎

Equations (3) and (14) have the same target and response function. Their different behavior comes from timing. A batch evaluates the same pre-update state and can overshoot it. Independent clocks continuously incorporate intervening revisions.

Figure 1 checks the result in the centered case. Panel (a) plots (12) in the sufficient statistics $m(\beta)$ and λ. The marked cases are $S = (1.5, .5)$ and $U = (1.5, 1)$. Panel (b) holds $\gamma = \beta k = 6$, $b/k = 1/2$, and $q_0 = .15$ fixed while changing only λ. Partial batches converge to $q_\beta = 1/2$, whereas full batches approach an alternating orbit. Direct iteration for 600 rounds on a parameter grid matches the strict analytic regions; the nonhyperbolic boundary is excluded from binary classification.

The benchmark has sharp limits. It is nonatomic; finite populations produce a Markov chain around the mean path. Revisers observe the same aggregate state and use identical primitives. Independent signals, heterogeneous objectives, delayed observations, and correlated random seeds require additional state variables. The parameter β measures logit precision only. A more capable system could instead change payoff estimates, forecasting horizons, or coordination mechanisms. Nonlinear

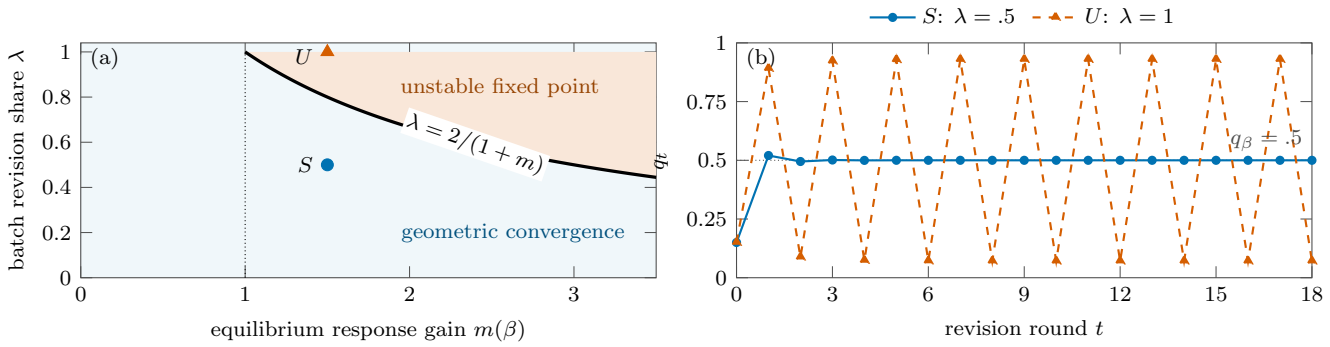


Fig. 1. Exact synchronization budget and matched trajectories. Panel (a) shows geometric stability below $\lambda = 2/[1 + m(\beta)]$ and fixed-point instability above it; equality is nonhyperbolic. Panel (b) holds $\beta k = 6$, $b/k = 1/2$, and $q_0 = 0.15$ fixed. A batch share $\lambda = .5$ converges, whereas $\lambda = 1$ approaches an alternating orbit.

congestion can also introduce multiple fixed points or other bifurcations. Finally, the theorem is positive, not a policy prescription: deliberate staggering can stabilize this model, but real systems must trade delay against other costs.

The benchmark nevertheless gives falsifiable extensions. In controlled populations of software agents, varying decision temperature and batch size while holding payoffs fixed should produce a flip boundary predicted by (9). Replacing common batches with independent exponential clocks should remove alternating instability while preserving the empirical fixed point. Failure would identify omitted mechanisms rather than validate a general claim about intelligence.

V. Conclusion

Sharper individual response need not imply smoother collective adjustment. In a binary congestion game, the static logit fixed point is unaffected by the fraction revising in a batch, yet its dynamic stability obeys an exact synchronization budget. Under an interior payoff crossing, equilibrium response gain rises monotonically with precision, full-batch updating crosses a unique flip threshold, and a two-cycle exists above that threshold. Independent-clock revision remains globally stable at every finite precision. Evaluations of automated multiagent systems should therefore report not only decision quality, but also the revision protocol that couples individually reasonable actions into a population trajectory.

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